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A Structured Optimal Route Policy*

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A FORWARD FORMULATION PROBLEM

Consider the following functional equation for the problem of getting from any point $t \in [0, T]$ to T ($\neq 0$ or ∞) at a minimal cost, when $c(t, u)$ is the cost of moving from t to u in one step, and $u \in [t, T]$.

$$n \geqslant 2: \qquad f_n(t) = \inf_{u \in [t,T]} \{c(t, u) + f_{n-1}(u)\}, \qquad t \in [0, T), \tag{1}$$

$$f_1(t) = c(t, T), \qquad t \in [0, T); \tag{2}$$

$$n \geqslant 1: \qquad f_n(T) = 0. \tag{3}$$

$f_n(t)$ has the interpretation of moving from t to T at an infimal total cost using at most n steps. We have the following lemma:

LEMMA 1. *If*

(i) *$c(t, u)$ is convex and bounded above in the region $t \in [0, T)$, $u \in [t, T]$;*

(ii) *$c(t, u) = c + k(t, u)$, for $t \in [0, T)$, $u \in [t, T]$, with $c > 0$, and $k(t, u)$ continuous in u, and in t, for $t \in [0, T)$, $u \in [t, T]$;*

(iii) *$k(t, u) \geqslant 0$, $k(u, u) = 0$, $t \in [0, T)$, $u \in [t, T]$;*

(iv) *$c(T, T) = 0$;*

then the sequence $\{f_n(\cdot)\}$ converges to a convex bounded function, $t \in [0, T)$, which is a unique solution of the equations

$$f(t) = \min_{u \in [t,T]} \{c(t, u) + f(u)\}, \; t \in [0, T], \tag{4}$$

$$f(T) = 0, \tag{5}$$

and hence an optimal policy $u(\cdot)$ exists for $t \in [0, T]$ for the routing problem of minimising the cost of getting from t of T.

* Notes in Decision Theory, Note Number 25.

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Proof. $f_1(\cdot)$ is convex, $t \in [0, T)$. (1) is clearly also true when $t = T$. We use (1) for $t \in [0, T)$. From (1), with $n \geq 2$, we have, if $r, s \in [0, T)$, since $\lambda r + (1-\lambda)s \in [0, T)$, if $0 \leq \lambda \leq 1$,

$$
\begin{aligned}
f_n(\lambda r + (1-\lambda)s) &= \inf_{u \in [\lambda r + (1-\lambda)s, T]} \{c(\lambda r + (1-\lambda)s, u) + f_{n-1}(u)\} \\
&\leq \inf_{\substack{u = \lambda p + (1-\lambda)q \\ p \in [r, T] \\ q \in [s, T]}} \{c(\lambda r + (1-\lambda)s, \lambda p + (1-\lambda)q) \\
&\qquad + f_{n-1}(\lambda p + (1-\lambda)q\} \leq \text{(using induction on } (n-1)) \\
&\qquad \lambda \inf_{p \in [r, T]} \{c(r, p) + f_{n-1}(p)\} + (1-\lambda) \inf_{q \in [s, T]} \{c(s, q) + f_{n-1}(q)\} \\
&= \lambda f_n(r) + (1-\lambda) f_n(s)
\end{aligned}
\tag{6}
$$

and hence $f_n(\cdot)$ is convex, $t \in [0, T)$.

It is easily seen that $\{f_n(\cdot)\}$ is monotonic decreasing in n, bounded above, and bounded below by 0, and hence converges to a function $f(\cdot)$. Clearly $\{f_n(\cdot)\}$ must converge in a finite number of steps, since $f_n(t) \leq c(t, T)$, $t \in [0, T]$, and if (1) required m steps to get to T, we would have $f_n(t) \geq mc \to \infty$ as $m \to \infty$. Hence m must be bounded above ($\leq$ some $\bar{m}$, $t \in [0, T]$), and thus convergence is uniform, $t \in [0, T]$. Hence $f(\cdot)$ satisfies (5), and (4) with "inf" instead of "min". Convergence and convexity of $\{f_n(\cdot)\}$ guarantees convexity of $f(\cdot)$. Convexity of $f(\cdot)$ and $c(t, \cdot)$ guarantees continuity on $(0, T)$ (Rockafellar [1, p. 82]). The conditions of the lemma guarantee that if u is close enough to T, then $c(t, T) \leq c(t, u) + f(u)$, $u \in [t, T]$, and that, with $t \neq 0$, together with the above continuity result, a minimiser exists for u. Hence, for $t \neq 0$, we can replace "inf" by "min". When $t = 0$, $\exists \bar{u} \in (0, T]$, such that if $u_1 \in [0, \bar{u}]$, $u_2 \in [u_1, T]$, then $c(0, u_2) < c(0, u_1) + c(u_1, u_2)$ and hence $u(0) \in [\bar{u}, T]$. Hence, combining this with the above results, we see that once again a minimising u exists. Hence we have "min" instead of "inf" as required for (4), and the specified optimal policy exists. The uniqueness of a solution to (4) is well known (see White [2, p. 62]). ∎

Note that the condition $C(T, T) = 0$ merely enables (1) and (4) to be true for $t = T$. It has no other role, although it is natural for optimal route problems.

Let us now define, $\forall$ feasible t, u, δ, $\nabla(t, u, \delta) = c(t, u + \delta) - c(t, u)$, $\nabla(u, \delta) = f(u + \delta) - f(u)$, $t \in [0, T)$, $u \in [t, T]$, $u + \delta \in [t, T]$. We then have the following theorem.

THEOREM 1. *If* $\tau \in [0, T)$, $t \in [0, \tau] \to \nabla(t, u, \delta) \geq \nabla(\tau, u, \delta)$, *and if the conditions of Lemma* 1 *apply, then an optimal policy* $u(\cdot)$ *exists with* $\tau \in [0, T]$, $t \in [0, \tau] \to u(\tau) \in [u(t), T]$. *If* $c(t, u)$ *is strictly convex in* (t, u), $t \in [0, T)$, $u \in (t, T]$, *then such a policy is unique.*

Proof. If $\tau = T$, the result is obvious. Now let $\tau \neq T$, and $t \in [0, \tau]$. Then

$$f(t) = \min_{u \in [t,T]} \{c(t, u) + f(u)\}, \tag{7}$$

$$f(\tau) = \min_{u \in [\tau,T]} \{c(\tau, u) + f(u)\}. \tag{8}$$

If $\{u(t)\}$ are the minimisers for a given t, the previous analysis indicates that $w(t) = \min\{u(t)\}$ exists. If $c(\cdot, \cdot)$ is strictly convex, a re-run of the convexity analysis shows that $c(t, u) + f(u)$ is strictly convex in u, and hence $u(t)$ is unique, $t \in [0, T)$, and hence in $[0, T]$. Now if $w(t) \in [0, \tau]$ our result follows. If $w(t) \in (\tau, T]$ the condition $\nabla(t, u, \delta) \geqslant \nabla(\tau, u, \delta)$ implies that the function $c(t, u) + f(u)$ reaches a minimum at least as early as does the function $c(\tau, u) + f(u)$, $u \in [\tau, T]$, because $\nabla(t, u, \delta) + \nabla(u, \delta) \leqslant 0 \rightarrow \nabla(\tau, u, \delta) + \nabla(u, \delta) \leqslant 0$, and the functions are convex. Hence $\tau \in [0, T]$, $t \in [0, \tau] \rightarrow w(\tau) \in [w(t), T]$. ▮

COROLLARY 1.1. *If $\tau \in [0, T]$, $t \in [0, \tau]$, then $\exists$ optimal sequence of moves from t which requires at least as many moves as does some optimal sequence from τ. Under the strict convexity conditions on $c(t, \tau)$, the optimal sequences are unique.*

Proof. Let t_0 $(= t)$, $t_1, t_2, \ldots, t_n$ $(= T)$ be an optimal sequence of moves from t_0, under $w(\cdot)$. Then $w(t_m) = t_{m+1}$, $0 \leqslant m \leqslant n - 1$. Then, from the theorem, if τ_0 $(= \tau)$, $\tau_1, \tau_2, \ldots, \tau_l$ $(= T)$ are optimal moves from τ_0 under $w(\cdot)$, we have $\tau_1 \in [t_1, T]$, $\tau_2 \in [t_2, T] \cdots \tau_l \in [t_m\ (= T), T]$. The result follows, since in the strictly convex case uniqueness has been established. ▮

A BACKWARD FORMULATION PROBLEM

Consider the analogous backward equations to (1), (2), (3), viz.

$$n \geqslant 2: \qquad F_n(t) = \inf_{u \in [0,t]} \{F_{n-1}(u) + c(u, t)\},\ t \in (0, T], \tag{9}$$

$$f_1(t) = c(0, t); \tag{10}$$

$$n \geqslant 1: \qquad F_n(0) = 0. \tag{11}$$

$F_n(t)$ has the interpretation of moving from 0 to t at an infimal total cost using at most n steps. We can easily prove the following lemma and theorem.

LEMMA 2. *If*

(i) *$c(u, t)$ is convex and bounded above in the region $t \in (0, T]$, $u \in [0, t]$;*

(ii) *$c(u, t) = c + k(u, t)$, for $t \in (0, T]$, $u \in [0, t]$, with $c > 0$ and $k(u, t)$ continuous in u and in t for $t \in (0, T]$, $u \in [0, t]$;*

(iii) $k(t, u) \geqslant 0$, $k(u, u) = 0$, $t \in (0, T)$, $u \in [0, t]$;

(iv) $c(0, 0) = 0$;

then the sequence $\{F_n(\cdot)\}$ *converges to a convex bounded function,* $t \in (0, T]$, *which is a unique solution of the equation*

$$F(t) = \min_{u \in [0,t]} \{F(u) + c(u, t)\}, \quad t \in [0, t] \tag{12}$$

$$F(0) = 0, \tag{13}$$

and hence an optimal policy $u(\cdot)$ *exists for* $t \in [0, T]$ *for the routing problem of minimising the cost of getting from* 0 *to* t.

Note that we do not need $c(0, 0) = 0$, but it is a natural presupposition for the real problem.

THEOREM 2. *If* $\tau \in (0, T]$, $t \in (0, \tau] \rightarrow \nabla(t, u, \delta) \geqslant \nabla(\tau, u, \delta)$, *and if the conditions of Lemma* 2 *apply, then an optimal policy* $u(\cdot)$ *exists with* $\tau \in (0, T]$, $t \in (0, \tau] \rightarrow u(t) \in [0, u(\tau)]$. *If* $c(u, t)$ *is strictly convex in* (t, u), $t \in [0, T]$, $u \in [0, t]$, *then such a policy is unique.*

COROLLARY 2.1. *If* $\tau \in (0, T]$, $t \in (0, \tau]$, *then there is an optimal sequence of moves from* 0 *to* τ *which requires at least as many moves as does some optimal sequence to* t. *Under the strict convexity conditions on* $c(u, t)$, *the optimal sequences are unique.*

AN APPLICATION

The general problem studied arose from an inventory control problem in which $c(t, u)$ is the cost of covering the demand in the period $[t, u]$ by a single order at t, when we have no inventory at t. The problem is one of minimising the total order and inventory holding costs over the period $[0, T]$.

REFERENCES

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